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Anisotropic dry friction and unilateral non-holonomic constraints☆

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ABSTRACT

An extension of the model of Coulomb's friction forces to the case of anisotropic friction, when the friction coefficients depend on the coordinates of the points of the contacting surfaces and their mutual orientation is proposed. The anisotropy can be due both to the properties of one of the contact surfaces (for example, the presence of "furrows" in the contact surface and their orientation), and to the corresponding properties of both contact surfaces and the position of the contact areas on the surfaces of the contacting bodies. When the properties of the surfaces of the interacting bodies are independent of the direction of the relative velocity and the mutual orientation of the bodies, the proposed model is identical with the classical Coulomb dry friction model. One of the models of anisotropic friction, which, in the limiting case when the friction force approaches infinity, reduces to a unilateral ideal unilateral constraint, is discussed.

Coulomb dry-friction forces have been used for a long time when describing the motions of mechanical systems. From the mathematical point of view, a feature of the model of dry-friction forces is the discontinuous form of the functions describing the interaction of solid bodies, which leads to difficulties when describing possible motions of the system and in the numerical modelling of the equations of motion. Some "paradoxes" and general properties of the motions of systems with dry friction were investigated in Refs. 1–10.

1. The construction of models of dissipative forces and the properties of the motions

Generalized forces in Lagrangian mechanics are said to be dissipative if their power is less than or equal to zero (the equality to zero must not be identical), i.e.

$$\sum_{i=1}^{k} Q_i(\dot{\mathbf{q}}, \mathbf{q}, t)\dot{q}_i \le 0$$

Here Q_i are generalized forces, and q_i and $\dot{q}_i$ are generalized coordinates and velocities ($i = 1, \ldots, k$). If, in a certain simply connected region of the generalized velocities, for any permissible values of the coordinates and time, the generalized forces have partial derivatives with respect to the generalized velocities and Cauchy's conditions

$$\frac{\partial Q_i}{\partial \dot{q}_j} = \frac{\partial Q_j}{\partial \dot{q}_i}, \quad i, j = 1, \ldots, k$$

are satisfied, then the differential form

$$\sum_{i=1}^{k} Q_i(\dot{\mathbf{q}}, \mathbf{q}, t)d\dot{q}_i = -d_{\dot{\mathbf{q}}}W(\dot{\mathbf{q}}, \mathbf{q}, t) \Rightarrow Q_i = -\frac{\partial W}{\partial \dot{q}_i},$$
$$i = 1, \ldots, k$$

If the number of degrees of freedom $k = 1$, the dissipative function $W(\dot{q}, q, t)$ exists and is equal to the primitive

$$W = -\int Q(\dot{q}, q, t)d\dot{q}$$

when calculating which the generalized coordinate q and time are assumed as the parameters.

We will consider the motion of a flat plate on a horizontal rough plane. Theorems on the change in the momentum of the plate and in the angular momentum about the centre of mass can be represented in the form

$$m\mathbf{w} = \mathbf{F}, \quad \mathbf{w} = (\ddot{X}_1, \ddot{X}_2), \quad J\ddot{\varphi} = M$$

where X_1 and X_2 are the coordinates of the centre of mass of the plate C in a fixed system of coordinates $OX_1X_2X_3$ connected with the plane, φ is the angle of rotation of the plate, J is the moment of inertia of the plate about the CX_3 axis, and $\mathbf{F}$ and M are the resultant friction forces and moment of the friction forces about the CX_3 axis.

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We will consider different methods of determining the quantities $\mathbf{F}$ and M. We will take as the basis of this determination the idea of a dissipative function, which models the properties of the interacting surfaces of the plate and the plane, for which we will consider a homogeneous function of the generalized velocities of order $n > 0$

$$W(\dot{\mathbf{q}}, \mathbf{q}) \geq 0, \quad W(\lambda\dot{\mathbf{q}}, \mathbf{q}) = |\lambda|^n W(\dot{\mathbf{q}}, \mathbf{q}), \quad \lambda \in \mathbf{R}^1,$$
$$\mathbf{q} = (X_1, X_2, \varphi)$$

We will assume that the dissipative function vanishes when and only when all the generalized velocities are equal to zero.

As was noted above, the generalized friction forces are produced by a dissipative function according to the rule

$$Q_i = -\frac{\partial W}{\partial \dot{q}_i}, \quad i = 1, 2, 3 \Rightarrow \sum_{i=1}^{3} Q_i \dot{q}_i = -nW(\dot{\mathbf{q}}, \mathbf{q}) \leq 0$$

As examples of dissipative functions, used when modelling mechanical systems, we note the case $n = 1$ (Coulomb dry friction), $n = 2$ (the dissipative Rayleigh function, and viscous friction which obeys the Stokes law), and $n = 3$ (the law of the drag of solid bodies moving in gases). Friction forces acting on a plate lead to a reduction in its kinetic energy in accordance with the theorem on its variation

$$\frac{dT}{dt} = -nW(\dot{X}_1, \dot{X}_2, \dot{\varphi}, X_1, X_2, \varphi) \leq 0,$$
$$T = \frac{1}{2}[m(\dot{X}_1^2 + \dot{X}_2^2) + J\dot{\varphi}^2] \tag{1.1}$$

The kinetic energy of the system is the square of the norm in a finite-dimensional velocity space. In a similar way the dissipative function generates an equivalent norm in velocity space and the following inequalities hold

$$c_1\sqrt{T} \leq W^{1/n} \leq c_2\sqrt{T}, \quad 0 < c_1 \leq c_2$$

using which, and relation (1.1), we obtain the inequalities

$$-nc_2^n T^{n/2} \leq \dot{T} \leq -nc_1^n T^{n/2} \tag{1.2}$$

Inequalities (1.2) enable us to judge the nature of the motion, namely, the way in which the kinetic energy of the system decreases.

When $0 < n < 2$ the following estimates hold

$$T^{1-n/2}(0) - n(1-n/2)c_2^n t \leq T^{1-n/2}(t) \leq T^{1-n/2}(0) - n(1-n/2)c_1^n t$$

from which it follows that the motion is completed not earlier than the instant of time t_1 and not later than the instant of time t_2, where

$$t_1 = \frac{1}{n(1-n/2)c_2^n}T^{1-n/2}(0), \quad t_2 = \frac{1}{n(1-n/2)c_1^2}T^{1-n/2}(0)$$

When $n = 2$ we obtain the following estimates for the kinetic energy of the system

$$T(0)\exp(-2c_2^2 t) \leq T(t) \leq T(0)\exp(-2c_1^2 t)$$

In this case the motion never stops, and the kinetic energy of the system remains confined between the two decreasing exponential functions.

Finally, when $n > 2$ we have the estimates

$$[T^{1-n/2}(0) + n(n/2-1)c_1^n t]^{1/(1-n/2)}$$
$$\leq T(t) \leq [T^{1-n/2}(0) + n(n/2-1)c_2^n t]^{1/(1-n/2)}$$

In this case, as in the case when $n = 2$, the motion never ceases and the kinetic energy decreases, remaining enclosed between the two decreasing rational-fractional functions.

The properties of the motions considered above remain true in the case of mechanical systems of general form with stationary constraints. The realization of any particular dissipation model depends on the specific model of the mechanical system considered and should be based on experimental results. It must not be assumed that the procedure for constructing dissipative forces, based on homogeneous dissipative functions, considered above is the only method of taking the interaction between contacting bodies into account.

For example, if a point moves over a rough horizontal circle, its equation of motion, taking into account the forces of dry friction, can be written in the form

$$\ddot{s} = -f\sqrt{g^2 + \dot{s}^4 r^{-2}}\,\mathrm{sign}\,\dot{s}$$

where $s(t)$ is the length of the arc of the circle of radius r traversed by the point, g is the acceleration due to gravity and f is the coefficient of the dry-friction forces. This example is not described within the framework of the scheme considered above. However, when there is no gravity, the dissipative function can be represented by a homogeneous function of order three, and as the radius of the circle approaches infinity the dissipative function is of the order of unity. Note that in this example one can construct an inhomogeneous dissipative function.

2. Anisotropic dry friction in the case of a plate of small dimensions

If the moment of inertia of the plate J is sufficiently large, and its contact area with the plane is sufficiently small and the moment of the friction force about the centre of mass of the plate is small, we can assume the angular velocity of rotation of the plate to be constant and consider it as equivalent to a point mass, moving along the plane under the action of the dry-friction force. Suppose the rough plane possesses anisotropy: the coefficients of dry friction are different for motion along the fixed OX_1 and OX_2 axes. These situations can be modelled by taking the dissipative function in the form

$$W(V_1, V_2) = FV, \quad V = [d_1V_1^2 + d_2V_2^2]^{1/2},$$
$$V_k = \dot{X}_k, \quad d_k > 0, \quad d_1 + d_2 = 1, \quad F > 0 \tag{2.1}$$

Another model of the dissipative forces is possible when

$$W(V_1, V_2) = F[d_1|V_1| + d_2|V_2|]$$

In the latter case the motion of the point can be represented by the direct product of one-dimensional motions, determined from the system of equations

$$m\dot{V}_k = -Fd_k\,\mathrm{sign}\,V_k \Rightarrow V_k(t) = V_k(0) - Fd_k t\,\mathrm{sign}\,V_k$$
$$X_k(t) = X_k(0) + V_k(0)t - \frac{1}{2}Fd_k t^2\,\mathrm{sign}\,V_k, \quad k = 1, 2 \tag{2.2}$$

In the (V_1, V_2) plane the trajectory of the representative point is a straight line, intersecting one of the coordinate axes. Motion will then occur in the direction of the origin of the system of coordinates along the coordinate axis with which the intersection of the straight line, defined in relations (2.2), occurred. The trajectory of motion in the OX_1X_2 plane consists of a curve in the first part and a rectilinear section in the second part, which ends in a stop.

In the case of the model of dry friction specified by the dissipative function (2.1), the equations of motion have the form

$$m\dot{V}_k = -\frac{d_k F V_k}{V}, \quad k = 1, 2 \Rightarrow \frac{\dot{V}_1}{d_1 V_1} = \frac{\dot{V}_2}{d_2 V_2}$$

Without loss of generality, we will assume $V_k \geq 0$, $\dot{V}_k \leq 0$. Then, the trajectory of the representative point in the (V_1, V_2) plane is described by the equation

$$V_1^{d_2} V_2^{d_1}(0) = V_2^{d_1} V_1^{d_2}(0)$$

Hence it follows that, in the general case, motions along the coordinate axes are completed simultaneously, when the representative point reaches the origin of coordinates. The time dependence of V_1 is found from the relation

$$\int\limits_{V_1(0)}^{V_1} \sqrt{1 + cV_1^{\alpha}}\, dV_1 = -\frac{F\sqrt{d_1}}{m} t, \quad \alpha = 2\left(\frac{d_2}{d_1} - 1\right) \tag{2.3}$$

where c is a constant, determined from the initial conditions. The integral in this relation is evaluated in explicit form if $d_1 = d_2 = 1/2$. In this case, the trajectory in the (V_1, V_2) plane is a straight line passing through the origin of coordinates, while the plane itself, from the point of view of the properties of the dry-friction forces, becomes isotropic. In general, the integral in (2.3) is evaluated by expanding the integrand in series. A characteristic feature of the forces of anisotropic dry friction, in the general situation is the fact that the friction forces do not coincide with the direction of the velocity of motion of the point, which leads to curving of the trajectory of motion. If the friction is isotropic ($d_1 = d_2 = 1/2$), the hodograph of the velocity and the trajectory of motion of the point become straight lines.

The anisotropy of the dry-friction forces can be explained by the properties of the surface of the plate, which is displaced over the plane. Using the hypotheses formulated above we will assume that the plate is rotated with constant angular velocity ω. We will introduce moving coordinates $Cx_1x_2x_3$, connected with the plate, and we will denote the projections of the velocity of the centre of mass onto the axis of the moving system of coordinates by v_1 and v_2. We will choose the dissipative function, which takes into account the anisotropy of the plate, by analogy with function (2.1), in the form

$$W(v_1, v_2) = Fv, \quad v = [d_1 v_1^2 + d_2 v_2^2]^{1/2}, \quad d_1 + d_2 = 1$$

We will represent the equations of motion of the mass centre in projections onto the moving axes in the form

$$\begin{aligned} m(\dot{v}_1 - \omega v_2) &= -\frac{\partial W}{\partial v_1} = -F d_1 v_1 v^{-1}, \\ m(\dot{v}_2 + \omega v_1) &= -\frac{\partial W}{\partial v_2} = -F d_2 v_2 v^{-1} \end{aligned} \tag{2.4}$$

The system of second-order differential equations is not integrable in explicit form. FromEq. (2.4) we have the equation

$$\frac{1}{2}\frac{d}{dt}(v_1^2 + v_2^2) = -Fm^{-1} v$$

which defines the change in the kinetic energy. A simple analysis, similar to that carried out in Section 1, enables us to conclude that the motion of the mass centre ends after a finite time interval.

In the model considered above it was assumed that the plate rotates with constant angular velocity, which is possible if we neglect the moment of the friction forces about the Cx_3 axis. This is justified when the dimensions of the contact area of the plate are small and the moment of rotational friction is correspondingly small. In this case we can assume that the angular velocity of rotation of the plate is almost constant in the time interval until the mass centre of the plate stops.

3. A model of anisotropic friction for a plate of finite dimensions

We will consider the simplest generalization of the models, presented in Section 2, in the case when the translational and rotational motions of the plate are taken into account.

Suppose the property of anisotropy of the friction is due to the fixed plane on which the plate of finite dimensions moves. This means that the coefficients of friction are different, for example, along the OX_1 and OX_2 axes due to the presence of “furrows” in the plane. If a uniform rough plate of finite dimensions moves over this plane, the dissipative function of the dry-friction forces can be represented in the form

$$W(\dot{\mathbf{q}}, \mathbf{q}) = \iint\limits_D F(\mathbf{r}, \mathbf{q})[d_1(\mathbf{E}_1, \mathbf{V})^2 + d_2(\mathbf{E}_2, \mathbf{V})^2]^{1/2} dx_1 dx_2$$

$$\mathbf{r} \in D, \ \mathbf{q} = (X_1, X_2, \varphi), \ \mathbf{V} = \dot{\mathbf{R}}_C + \dot{\varphi}[\mathbf{E}_3 \times \Gamma_3(\varphi)\mathbf{r}],$$

$$\mathbf{R}_C = (X_1, X_2), \ d_1 + d_2 = 1$$

$$\Gamma_3(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{3.1}$$

Here D is the region occupied by the plate, $\mathbf{r}$ is the radius-vector of a point of the plate in the moving system of coordinates $Cx_1x_2x_3$ connected with the plate with the origin in its mass centre, $\mathbf{E}_k$ is the unit vector of the fixed system of coordinates $OX_1X_2X_3$, $\Gamma_3(\varphi)$ is the orthogonal operator of rotation of the plate by an angle φ with respect to the Cx_3 axis, and $\mathbf{R}_C$ is the radius-vector of the mass centre of the plate with respect to the fixed system of coordinates $OX_1X_2X_3$. The function $F(\mathbf{r}, \mathbf{q})$ is positive, has an upper bound and defines the specific value of the dry-friction force. The coefficients d_1 and d_2 are constant when the characteristics of the dry friction of the plane are independent of the position of the point in the plane. The friction becomes Coulomb friction when $d_1 = d_2 = 1/2$.

Note that in the model considered we must give up the model of an absolutely rigid body for the plate, since, in general, the surface of the rigid body may be in contact with the plane at only three points, while in the model considered above it was assumed that contact occurs over a certain region D with a pressure, the value of which depends on the coordinates of the points of contact and other kinematic characteristics of the motion (for example, the contact of a pneumatic tyre with the road coating). Another way of determining the specific friction force when two bodies are in contact is based on the use of the Hertz model of the contact between two elastic bodies and the corresponding normal pressure distribution.[5]

If the anisotropy of the dry-friction forces is due to the plate, while the plane along which the plate moves is isotropic, then, using the projections of the velocities of points of the plate onto the axes of the moving system of coordinates $Cx_1x_2x_3$, we can be represent the dissipative function in a form similar to (3.1)

$$W(\dot{\mathbf{q}}, \mathbf{q}) = \iint\limits_D F(\mathbf{r}, \mathbf{q})[d_1(\mathbf{e}_1, \Gamma_3(-\varphi)\mathbf{V})^2 + d_2(\mathbf{e}_2,$$

$$\Gamma_3(-\varphi)\mathbf{V})^2]^{1/2} dx_1 dx_2, \quad d_1 + d_2 = 1 \tag{3.2}$$

Here $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors of the moving system of coordinates.

Model (3.1) can be used when describing the slippage of an isotropic tyre (a "bald" protector) on asphalt with anisotropic friction, which occurs due to the longitudinal "furrows" on the road when it is being repaired. The model of anisotropic friction (3.2) is suitable when describing the slippage of a tyre with a non-worn protector (the anisotropy of the friction is due to the pattern of the protector) along a road, the surface of which is isotropic.

4. Slippage of a "motorcycle" on a plane with anisotropic friction

We will consider, as an example, the simplest model of a "motorcycle" with locked wheels, when its plane remains orthogonal to the horizontal plane, along which it moves. In fact we are dealing with the motion of a plate, which is in contact with a rough plane at two points. We will assume that the points of contact with the plane have coordinates $(\pm l, 0)$ in the Cx_1x_2 system of coordinates, connected with the motorcycle, where C is the mass centre of the motorcycle and the third coordinate of the mass centre is much less than l. In this case we will neglect the effect of the acceleration of the mass centre on the redistribution of the normal reactions at the two points of contact of the front and rear wheels. The front wheel of the motorcycle is turned through an angle ϑ, which does not change during the motion.

4.1. The first case

We will assume that the anisotropy of the friction is due to the road and not due to the treads of the motorcycle wheels. We will use the model of the dissipative function (3.1), the integral in which reduces to the sum of two terms and can be represented in the form

$$W(\dot{X}_1, \dot{X}_2, \dot{X}_3, \varphi) = F(V_1 + V_2), \quad \dot{X}_3 = l\dot{\varphi}$$

$$V_n = \sqrt{d_1[\dot{X}_1 + (-1)^n \dot{X}_3 \sin\varphi]^2 + d_2[\dot{X}_2 - (-1)^n \dot{X}_3 \cos\varphi]^2},$$

$$n = 1, 2 \tag{4.1}$$

Here X_1 and X_2 are the coordinates of the mass centre of the motorcycle in the fixed system of coordinates OX_1X_2, along the axis of which the property of dry-friction anisotropy manifests itself, φ is the angle between the longitudinal axis of the motorcycle and the OX_1 axis, and F is the specific value of the dry-friction force at the points of contact of the motorcycle wheels with the road. The dissipative function (4.1) is independent of the angle of rotation of the front wheel.

The equations of motion, represented by the theorems on the change in the momentum and in the angular momentum about the mass centre, have the form

$$m_k \ddot{X}_k = -F\left(\frac{\partial V_1}{\partial \dot{X}_k} + \frac{\partial V_2}{\partial \dot{X}_k}\right), \quad k = 1, 2, 3, \quad m_1 = m_2 = m,$$

$$m_3 = Jl^{-2} \tag{4.2}$$

where m and J are the mass and moment of inertia of the motorcycle about the mass centre. Equation (4.2) allow of the simplest classes of motions: longitudinal motion and rotation about a fixed mass centre. The longitudinal motion, when $\dot{X}_{3=0}$, exists for any value of the angle φ, since the right-hand side of the third equation of system (4.2) then vanishes, and it is described by equations similar to Eq. (2.2),

$$m\ddot{X}_k = -2Fd_k\dot{X}_kV^{-1}, \quad V = \sqrt{d_1\dot{X}_1^2 + d_2\dot{X}_2^2}, \quad k = 1, 2$$

Rotation around the fixed mass centre also always exists and described by the equation

$$J\ddot{\varphi} = -2Fl\sqrt{d_1\sin^2\varphi + d_2\cos^2\varphi}\,\mathrm{sign}\,\dot{\varphi}$$

The rotation ceases after a finite time interval.

In the general case, the translational and rotational motion of the motorcycle affect one another and are determined by the solutions of the system of non-linear equation (4.2)

4.2. The second case

If the anisotropy of the dry friction is due to the properties of the tread of the motorcycle wheels, the dissipative function is taken as that given by expression (3.2), when the integral on the right-hand side is replaced by the sum of two terms. Using the projection of the velocities of the points of contact onto the moving axes Cx_1x_2, we can represent the dissipative function in the form

$$W(v_1, v_2, v_3) = F(V_1 + V_2) \tag{4.3}$$

where

$$V_1 = \sqrt{(1 - k\cos 2\vartheta)v_1^2 + (1 + k\cos 2\vartheta)(v_2 + v_3)^2 - 2kv_1(v_2 + v_3)\sin 2\vartheta}$$

$$V_2 = \sqrt{(1-k)v_1^2 + (1+k)(v_2 - v_3)^2}; \quad v_3 = l\dot{\varphi}, \quad 0 < k < 1 \tag{4.4}$$

It is assumed that the characteristics of the friction forces are the same for the front and rear wheels of the motorcycle. Here v_1 and v_2 are the projections of the velocity of the mass centre of the motorcycle onto its longitudinal and transverse axes, connected with the motorcycle and rotating with an angular velocity $\dot{\varphi}$, while ϑ is the fixed angle of rotation of the front wheel.

The equations of motion, represented by theorems on the change in the momentum and in the angular momentum about the mass centre, have the form (see Eq. (2.4))

$$m(\dot{v}_1 - l^{-1}v_3v_2) = -\frac{\partial W}{\partial v_1}, \quad m(\dot{v}_2 + l^{-1}v_3v_1) = -\frac{\partial W}{\partial v_2},$$

$$Jl^{-2}\dot{v}_3 = -\frac{\partial W}{\partial v_3} \tag{4.5}$$

The system of equation (4.2), according to the discussion in Section 1, has solutions which approach zero after a finite time. Particular solutions of the equations, corresponding to certain initial conditions, can be investigated by numerical methods.

We will investigate the simplest types of motions of the motorcycle with locked wheels: translational motion and rotation about the mass centre. In the first case the angle φ is constant ($v_3 = 0$), and from the third equation of system (4.5) we have the equality

$$\frac{\partial V_1}{\partial \upsilon_3} + \frac{\partial V_2}{\partial \upsilon_3} = 0 \Rightarrow V_1^{-1}[(1+k\cos 2\vartheta)\upsilon_2 - k\upsilon_1 \sin 2\vartheta] = V_2^{-1}(1+k)\upsilon_2 \tag{4.6}$$

The functions v_1 and v_2 are defined by Eq. (4.4) when $v_3 = 0$.

The conditions (4.6) are satisfied for any v_1 and v_2 if the motorcycle wheels are in the single plane $\vartheta = 0, \pi$. In this case translational motions of the motorcycle exist, described by the first two equations of system (4.5):

$$m\dot{\upsilon}_1 = -2FV^{-1}(1-k)\upsilon_1, \; m\dot{\upsilon}_2 = -2FV^{-1}(1+k)\upsilon_2, \quad V = \sqrt{(1-k)\upsilon_1^2 + (1+k)\upsilon_2^2} \tag{4.7}$$

which are similar to the corresponding equations in Section 2. In the case of an arbitrary angle of rotation of the front wheel, the necessary condition for translational motions (4.6) to exist is represented by an equation in the ratio v_1/v_2. Translational motion will exist if the value of the ratio obtained from Eq. (4.6) is retained by virtue of the first two equations of system (4.5).

In the case of rotation around the Cx_3 axis, the relations $v_1 = v_2 = 0$, $v_3 \neq 0$ hold, for which the right-hand sides of the first two equations of system (4.4) must be identically equal to zero:

$$V_1^{-1}\upsilon_3 k \sin 2\vartheta = 0, \quad [V_1^{-1}(1+k\sin 2\vartheta) - V_2^{-1}(1+k)]\upsilon_3 = 0$$

$$V_1 = \sqrt{(1+k\cos 2\vartheta)\upsilon_3^2}, \quad V_2 = \sqrt{(1+k)\upsilon_3^2} \Rightarrow \sin 2\vartheta = 0,$$

$$\sqrt{1+k\cos 2\vartheta} = \sqrt{1+k}$$

The last conditions will hold if the angle of rotation of the front wheel $\vartheta = 0, \pi$. The angle of rotation of the motorcycle in this case is found from the equation

$$J\ddot{\varphi} = -2Fl\sqrt{1+k}\,\mathrm{sign}\,\dot{\varphi} \Rightarrow \varphi(t) = \varphi(0) + \dot{\varphi}(0)t - Flt^2\sqrt{1+k}\,\mathrm{sign}\,\dot{\varphi}$$

The motorcycle rotates uniformly slowly and stops after a finite time.

5. Unilateral non-holonomic constraints

We will consider the well-known problem of the Chaplygin sleigh, when the following bilateral non-holonomic constraint

$$-\dot{x}\sin\varphi + \dot{y}\cos\varphi = 0 \tag{5.1}$$

is imposed on the displacement of the mass centre of a rigid body, performing plane-parallel motion. Here x, y and φ are the coordinates of the mass centre of the sleigh and its angle of rotation about the fixed OX axis. We will assume that at the contact point the shape of the blade of the sleigh can be represented by a hemisphere, the plane of cross-section of which is orthogonal to the horizontal OXY plane, on which slippage of the sleigh occurs. In this case it is logical to assume that the sleigh can slide along the blades and is displaced sideways, coinciding with the convexity of the hemisphere, while the rate of displacement of the contact point in the opposite direction, corresponding to the sharp edge of the hemisphere, is equal to zero. In fact, we are dealing with a sleigh in which one of the sides of the blade is sharp, while the other is blunted and does not shift the sleigh sideways. Then, a unilateral non-holonomic constraint occurs which is described by an inequality which arises from relation (5.1):[11]

$$-\dot{x}\sin\varphi + \dot{y}\cos\varphi \geq 0 \tag{5.2}$$

The angular velocity of rotation of the sleigh remains constant during the whole motion, and the trajectory of the mass centre consists of two parts:[11] uniform motion over the section of a straight line with a change to uniform motion in a circle. The first stage of the motion exists if, at the initial instant of time, we have a strict inequality in relation (5.2), while the transition to the second stage corresponds to the vanishing of the left-hand side of inequality (5.2), which remains equal to zero during the whole of the second stage (the classical problem of a Chaplygin sleigh).

We will assume that constraint (5.2) is not imposed on the system, and dissipative forces act, which generate the dissipative function

$$W = \frac{1}{2}mb(|\upsilon_2| - \upsilon_2), \quad \upsilon_2 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi, \quad b > 0$$

We will represent the equations of motion of the system, according to Eq. (2.4), in the form

$$\dot{\upsilon}_1 - \omega\upsilon_2 = 0, \quad \dot{\upsilon}_2 + \omega\upsilon_1 = \Psi(\upsilon_2), \quad \omega = \dot{\varphi} = \mathrm{const}$$

$$\Psi(\upsilon_2) = \begin{cases} b, & \upsilon_2 < 0 \\ b\xi, \quad 0 \leq \xi \leq 1, & \upsilon_2 = 0 \\ 0, & \upsilon_2 > 0 \end{cases} \tag{5.3}$$

the function $-\Psi(v_2)$ for non-zero values of the argument is equal to the derivative of the dissipative function $W(v_2)$, divided by the mass. The function $\Psi(v_2)$ can take any values in the interfval $[0, b]$ for a zero value of the argument.

The model of the friction presented can be regarded as "unilateral" anisotropic dry friction.If $v_1(0) \leq b/\omega$ at the initial instant, the motion of the sleigh, according to Eq. (5.3), can occur in accordance with three scenarios.

5.1. *The first case*

If $v_2(0) = 0$ at the initial instant of time, system (5.3) has the solution

$$\upsilon_1(t) = \upsilon_1(0) = b\xi/\omega = \Psi(0), \quad \upsilon_2(t) = 0$$

This motion is identical with the motion of a Chaplygin sleigh, when the mass centre describes a circle with constant velocity.

5.2. *The second case*

If $v_2(0) > 0$, the system of equation (5.3) can be represented in the form

$$\dot{z} + i\omega z = 0, \quad z = \upsilon_1 + i\upsilon_2 \Rightarrow z(t) = \upsilon\exp[i(\alpha - \omega t)]$$
$$\upsilon_1(0) = \upsilon\cos\alpha, \quad \upsilon_2(0) = \upsilon\sin\alpha, \quad \upsilon > 0, \quad 0 < \alpha < \pi \tag{5.4}$$

According to solution (5.4), the system moves as a plate on a smooth plane with constant velocity of the mass centre and constant angular velocity. This motion occurs in the time interval $(0, t_1)$, $t_1 = \alpha/\omega$. At the instant of time t_1 the component of the velocity $v_2(t_1)$ vanishes, and further motion occurs according to the first case.

5.3. The third case

If $v_2(0) < 0$, the system of equation (5.3) can be represented in the form

$$\dot{z} + i\omega z = b \Rightarrow z(t) = [1 - \exp(-i\omega t)]b/\omega + \upsilon\exp[i(\beta - \omega t)]$$
$$\upsilon_1(0) = \upsilon\cos\beta, \quad \upsilon_2(0) = \upsilon\sin\beta, \quad \upsilon > 0, \quad -\pi < \beta < 0 \tag{5.5}$$

If $b\omega^{-1} \gg v$, the component of the velocity

$$\upsilon_2(t) = \mathrm{Im}z(t) = \upsilon\sin(\beta - \omega t) + b\omega^{-1}\sin(\omega t)$$

vanishes fairly rapidly, and further motion will occur according to the first case. The value of the instant of time when the transverse component of the velocity of the mass centre of the sleigh vanishes is described by the quantity

$$t_2 \approx -\upsilon\omega b^{-1}\sin\beta$$

In the limit, when the parameter b, which defines the value of the friction force, approaches infinity, the unilateral non-holonomic constraint (5.2) is obtained.

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